STRESS FIELDS CLOSE TO THE BOUNDARY OF A STOCHASTICALLY INHOMOGENEOUS HALF-PLANE DURING CREEP

N. N. Popov and Yu. P. Samarin

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A structurally inhomogeneous material makes conditions for development of a number of mechanical effects which cannot be studied within the framework of classical phenomenological theories; one of them is the effect of a boundary layer. It consists of the fact that close to the boundary of a body with structural inhomogeneity there is a boundary layer in which the stress—strained state differs from that in the internal regions. At the boundary of the body, stress concentrations arise which may reach considerable values. The effect of a boundary layer for elastic bodies has been studied in [1-6]. In creep theory there is a well-known solution of stochastic problems for internal regions quite distant from the boundary of the body [7-10], but the effect of stochastic inhomogeneities close to the surface at which boundary conditions are prescribed has not previously been studied.

In this work a boundary problem is considered for the stressed state of a randomly inhomogeneous half-plane under creep conditions. It has been established that there is a narrow boundary layer in which stress fluctuations may be much greater than in the depth of the half-plane. An example is provided for estimating the reliability of a stochastically inhomogeneous plate according to the failure criterion as a result of stress outbursts.

1. Let a load be applied to the boundary of a stochastically inhomogeneous half-plane $x_2 \geq 0$ which is under conditions of a plane stressed state,

$$\sigma_{22}|_{x_2=0} = \sigma_{22}^0 = \text{const},\ \sigma_{12}|_{x_2=0} = 0, \tag{1.1}$$

and stress σ_{11} satisfies the condition for macroscopic homogeneity $\langle\sigma_{11}\rangle = \sigma_{11}^0 = \text{const}$ which corresponds to application of stresses σ_{11}^0 constant along x_2 with $x_1 = \pm h$, where h is quite large.

The nonlinear boundary problem of creep is solved with an equation of state of the flow theory type in stochastic form [7]

$$\sigma_{ij,j} = 0,\ \Lambda_{ij}\Lambda_{kl}\varepsilon_{jk,il} = 0, \tag{1.2}$$
$$\varepsilon_{ij} = cs^n(\sigma_{ij} - (1/3)\delta_{ij}\sigma_{mm})[1 + \alpha U(x_1, x_2)]$$

with boundary conditions (1.1) (all of the indices take values of 1 and 2). Here ε_{ij} are strain tensor components; Λ_{jk} is elementary antisymmetrical pseudotensor; s is stress intensity; c and n are material constants; $U(x_1, x_2)$ is a random function describing material mechanical property fluctuation ($\langle U\rangle = 0$, $\langle U^2\rangle = 1$); α is a number playing the role of a mechanical property variation coefficient; δ_{ij} is Kronecker symbol. For repeated indices summation from 1 to 2 is carried out.

Stress tensor δ_{ij} taking account of (1.1) may be presented in the form of a sum of two terms

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^*,\ \langle\sigma_{ij}\rangle = \sigma_{ij}^0,\ \langle\sigma_{ij}^*\rangle = 0 \tag{1.3}$$

(σ_{ij}^* is stress fluctuation).

In [7] the nonlinear creep problem (1.1), (1.2) was linearized relative to fluctuations

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$$\sigma^*_{ij,j} = 0,$$
$$\sigma^*_{11,22}(2 + k_1p_1) + \sigma^*_{22,22}(-1 + k_1p_2) + \sigma^*_{11,11}(-1 + k_2p_1) + + \sigma^*_{22,11}(2 + k_2p_2) - 6\sigma^*_{12,12} = -\alpha\,(U_{,22}p_1 + U_{,11}p_2),$$
$$k_i = np_i/(2s^{02}),\ p_1 = 2\sigma^0_{11} - \sigma^0_{22},\ p_2 = 2\sigma^0_{22} - \sigma^0_{11},$$
$$s^{02} = \sigma^{02}_{11} + \sigma^{02}_{22} - \sigma^0_{11}\sigma^0_{22}. \tag{1.4}$$

Boundary conditions for system (1.4) taking account of (1.1) and (1.3) have the form

$$\sigma^*_{22}|_{x_2=0} = 0,\ \sigma^*_{12}|_{x_2=0} = 0. \tag{1.5}$$

Let homogeneous function $U(x_1, x_2)$, describing the material stochastic properties, be almost periodic and oscillate rapidly [1]:

$$U = \sum_{k=1}^{\infty} A^{(k)} \cos\left[\omega\left(\beta_1^{(k)}x_1 + \beta_2^{(k)}x_2\right) + \varphi^{(k)}\right],$$

where ω is a large parameter having a dimension inverse to length; $\beta_i^{(k)}$ are dimensionless values of the order of unity; $A^{(k)}$ and $\varphi^{(k)}$ are random values exhibiting properties: $\varphi^{(k)}$ has uniform distribution in the range $(0;\ 2\pi)$, $\langle A^{(k)}\rangle = 0$, $\langle A^{(k)}A^{(\ell)}\rangle = 0$ $(k \neq \ell)$, $\langle\varphi^{(k)}\varphi^{(l)}\rangle = 0$ $(k \neq \ell)$, $\langle A^{(k)}\varphi^{(l)}\rangle = 0$. If in addition it is assumed that all $A^{(k)}$ are distributed the same with finite dispersion, then according to the central limiting theorem, random function U may be assumed to be distributed by a normal rule.

For the convenience of computation it is desirable to change over to complex functions

$$U = \sum_{k=1}^{\infty} B^{(k)} \exp\left[i\omega\left(\beta_1^{(k)}x_1 + \beta_2^{(k)}x_2\right)\right]\left(B^{(k)} = A^{(k)}\exp\left(i\varphi^{(k)}\right)\right). \tag{1.6}$$

Solution of system (1.4) is presented as

$$\sigma^*_{ij} = \sum_{k=1}^{\infty}\left(v_{ij}^{(k)} + w_{ij}^{(k)}\right). \tag{1.7}$$

Here $v_{ij}^{(k)}$ is partial solution of system (1.4) obtained with substitution of function U by the k-th term of expansion (1.6); $w_{ij}^{(k)}$ is solution of a homogeneous system corresponding to (1.4) satisfying the conditions $w_{22}^{(k)} = -v_{22}^{(k)}$, $w_{12}^{(k)} = -v_{12}^{(k)}$ for $x_2 = 0$. In the future where this does not lead to misunderstandings the upper index k will be discarded. Functions $v_{k\ell}$ may be found in the form

$$v_{kl} = f_{kl}\exp\left[i\omega(\beta_1x_1 + \beta_2x_2)\right],\ f_{kl} = \text{const}. \tag{1.8}$$

After substituting (1.8) in (1.4) in order to find $f_{k\ell}$ a set of algebraic equations is obtained from which it follows that

$$f_{kl} = 2\alpha s^{02}\frac{B\left(\beta_k\beta_l - \delta_{kl}\beta^2\right)\left(p_1\beta_2^2 + p_2\beta_1^2\right)}{4s^{02}\beta^4 + n\left(p_1\beta_2^2 + p_2\beta_1^2\right)^2} \tag{1.9}$$
$$(\beta^2 = \beta_1^2 + \beta_2^2).$$

Series $\sum_{k=1}^{\infty} v_{ij}^{(k)}$ gives a solution at a distance from the boudnary of the half-plane without taking account of the boundary effect. Functions w_{ij} have the character of a boundary layer: they fade rapidly with distance from the boundary of the half-plane. We write them in the form

$$w_{ij} = g_{ij}(x_2)\exp(i\omega\beta_1x_1), \tag{1.10}$$

where function $g_{ij}(x_2)$ satisfies a set of normal differential equations

$$i\omega\beta_1g_{11} + g_{12,2} = 0,\ i\omega\beta_1g_{12} + g_{22,2} = 0,$$
$$g_{11,22}(2 + k_1p_1) + g_{22,22}(-1 + k_1p_2) - \omega^2\beta_1^2g_{11}(-1 + k_2p_1) - \omega^2\beta_1^2g_{22}(2 + k_2p_2) - 6i\omega\beta_1g_{12,2} = 0 \tag{1.11}$$

with boundary conditions

$$g_{22}|_{x_2=0} = -f_{22},\ g_{12}|_{x_2=0} = -f_{12}. \tag{1.12}$$

Set (1.11) was obtained by substituting (1.10) in the homogeneous set corresponding to (1.4).

Solution of set (1.11) with $\sigma^0_{11} \neq \sigma^0_{22}$ is written as

$$g_{ij} = \sum_{s=1}^{4} C_s X_{ij}^s \exp(r_s x_2).$$

Here C_s are arbitrary constants; X_{ij}^s are coordinates of the eigenvector corresponding to root r_s of a characteristic equation, which may lead to a biquadratic equation

$$(2 + k_1 p_1) r^4 - 2\omega^2 \beta_1^2 (2 + k_1 p_2) r^2 + \omega^4 \beta_1^4 (2 + k_2 p_2) = 0. \tag{1.13}$$

Of the four roots of (1.13) two of them have positive real parts. Since with $x_2 \to \infty$ the boundary effect should fade, all constants relating to these roots equal zero. In order to define the other two constants boundary conditions (1.12) are used.

2. Let in particular $\sigma_{11}^0 = \sigma_{22}^0 = \sigma^0$. Then roots of characteristic equation (1.13) will be $r_1 = r_2 = -\omega\beta_1$, $r_3 = r_4 = \omega\beta_1$. Solution of set (1.11), which is infinitely small with $x_2 \to \infty$, found by the method of indeterminate coefficients, has the form

$$g_{11} = (-iC_1 + iC_2 - iC_2\omega\beta_1 x_2)\exp(-\omega\beta_1 x_2), \tag{2.1}$$
$$g_{12} = (C_1 + C_2\omega\beta_1 x_2)\exp(-\omega\beta_1 x_2),$$
$$g_{22} = (iC_1 + iC_2 + iC_2\omega\beta_1 x_2)\exp(-\omega\beta_1 x_2).$$

By using boundary conditions (1.12) and expression (1.9) we find the integration constants:

$$C_1 = -2\alpha\sigma^0 \frac{B\beta_1\beta_2}{(4+n)\beta^2},\quad C_2 = 2\alpha\sigma^0 \frac{B(\beta_1\beta_2 - i\beta_1^2)}{(4+n)\beta^2}. \tag{2.2}$$

By substituting (1.8)-(1.10), (2.1), (2.2) in (1.7), solution of boundary problem (1.4), (1.5) is presented in the form

$$\sigma_{11}^* = \frac{2\alpha\sigma^0}{(4+n)} \sum_{k=1}^{\infty} \frac{B^{(k)}\exp(i\omega\beta_1^{(k)}x_1)}{\beta^{(k)2}} \{-\beta_2^{(k)2}\exp(i\omega\beta_2^{(k)}x_2) +$$
$$+ [\beta^{(k)2} + 2\beta_1^{(k)}\beta_2^{(k)}i - \omega\beta_1^{(k)2}x_2(\beta_1^{(k)} + \beta_2^{(k)}i)]\exp(-\omega\beta_1^{(k)}x_2)\},$$
$$\sigma_{22}^* = \frac{2\alpha\sigma^0}{(4+n)} \sum_{k=1}^{\infty} \frac{B^{(k)}\exp(i\omega\beta_1^{(k)}x_1)\beta_1^{(k)2}}{\beta^{(k)2}} \times$$
$$\times\{-\exp(i\omega\beta_2^{(k)}x_2) + [1 + \omega x_2(\beta_1^{(k)} + \beta_2^{(k)}i)]\exp(-\omega\beta_1^{(k)}x_2)\},$$
$$\sigma_{12}^* = \frac{2\alpha\sigma^0}{(4+n)} \sum_{k=1}^{\infty} \frac{B^{(k)}\exp(i\omega\beta_1^{(k)}x_1)\beta_1^{(k)}}{\beta^{(k)2}} \times$$
$$\times\{\beta_2^{(k)}\exp(i\omega\beta_2^{(k)}x_2) + [-\beta_2^{(k)} + \omega\beta_1^{(k)}x_2(-\beta_1^{(k)}i + \beta_2^{(k)})]\exp(-\omega\beta_1^{(k)}x_2)\}.$$

The dispersion of random stress field $D_{ij} = \langle|\sigma_{ij}^*|^2\rangle$ is calculated with the condition that all values of $\beta_1^{(k)}$ equal unity. Random field U, prescribed by expansion (1.6), may be assumed to be approximately isotropic [1]. Taking account of the conditions imposed on values of $A^{(k)}$, $\varphi^{(k)}$, and $\langle U^2\rangle = 1$, the dispersion of the random stress field is expressed as follows:

$$D_{11}(x_2) = \frac{2\alpha^2\sigma^{02}}{(4+n)^2}\{(5 - 6\omega x_2 + 2\omega^2 x_2^2)\exp(-2\omega x_2) - \tag{2.3}$$
$$- 2[(1 - \omega x_2)\cos\omega x_2 + (2 - \omega x_2)\sin\omega x_2]\exp(-\omega x_2) + 1\},$$
$$D_{22}(x_2) = \frac{2\alpha^2\sigma^{02}}{(4+n)^2}\{(2\omega^2 x_2^2 + 2\omega x_2 + 1)\exp(-2\omega x_2) -$$
$$- 2[(1 + \omega x_2)\cos\omega x_2 + \omega x_2\sin\omega x_2]\exp(-\omega x_2) + 1\},$$
$$D_{12}(x_2) = \frac{2\alpha^2\sigma^{02}}{(4+n)^2}\{(2\omega^2 x_2^2 - 2\omega x_2 + 1)\exp(-2\omega x_2) -$$
$$- 2[(1 - \omega x_2)\cos\omega x_2 + \omega x_2\sin\omega x_2]\exp(-\omega x_2) + 1\}.$$

3. The results obtained make it possible to analyze the main features of the effect of a boundary layer during creep.

At the boundary of half-plane $x_2 = 0$ the first equation of (2.3) gives

$$D_{11}(0) = 8\alpha^2\sigma^{02}/(4+n)^2, \tag{3.1}$$

and with $x_2 \to \infty$, $D_{11}(\infty) = 2\alpha^2\sigma^{02}/(4 + n)^2$. Stress concentration, arising at the boundary of the half-plane $x_2 = 0$, is characterized by the square root of the ratio of these expressions:

$\rho = \sqrt{D_{11}(0)/D_{11}(\infty)} = 2$. For an elastic half-plane this value with Poisson's ration $\nu = 0.25$, calculated from results in [2], is somewhat less ($\rho = 1.55$).

Presented in Fig. 1 is the dependence of normalized dispersions of stresses D^0_{ij} ($D^0_{ij} = D_{ij}(x_2)/D_{ij}(\infty)$) on dimensionless coordinates ωx_2. Normalized dispersions D^0_{12}, D^0_{22} at the half-plane boundary equal zero, and $D^0_{11} = 4$. With an increase in x_2, dispersions quite rapidly approach constant values coinciding with their values for unbounded material. With $\omega x_2 \geq 5$ the relative error from substituting normalized dispersions by unity does not exceed 5%. Therefore, it is possible to assume that the zone of the boundary effect has a width of the order of $5/\omega$ (ω is frequency of microinhomogeneity fluctuation).

Given in Table 1 in relation to variables α and n are values of coefficient of variation d_{11} at the boundary of the half-plane $x_2 = 0$. Here d_{11} takes maximum values which exceed by a factor of two the corresponding values at infinity. Coefficients of variation d_{12} and d_{22} with fixed α and n take maximum values at the same points as for normalized dispersions D^0_{12} and D^0_{22}, i.e., in the boundary layer. They are greater by factors of 1.05 and 1.2 than their asymptotic values respectively.

It is well known that the power rule for creep (1.2) is a good description of material behavior only in a small area of change in stresses, and a better result is given by the hyperbolic sine rule. However, in a comparatively small interval of change in stresses the hyperbolic sine rule may be approximated by a power function, and then with small stresses $n \approx 0$, and with large stresses n = 5-7. Therefore, in the region of large stresses the value of d_{11} for metallic materials is found within the limits from 1.29($\alpha = 0.05$) to 12.86% ($\alpha = 0.5$), and with small stresses it changes from 3.53 ($\alpha = 0.05$) to 35.36% ($\alpha = 0.5$). Thus, in the surface layer the coefficient of variation for stress fluctuation reaches considerable values, which may be greater than for deep layers.

Whence it is clear that stress fluctuations in the boundary layer play a considerable role in resolving the question of structural reliability according to the stress-rupture strength criterion and instantaneous local failures as a result of stress outbursts. Nonconsideration of boundary effects may lead to an unfounded high estimate of the operating efficiency of structures under creep conditions.

4. The solution obtained for the stochastic boundary problem of creep may be applied for approximate estimation of the reliability of a randomly homogeneous flat plate under conditions of biaxial uniform tension (a similar problem without taking account of boundary effects was considered in [11]). Random function $U(x_1, x_2)$, describing the behavior of material rheological characteristics, is assumed to be normal. In this way stress tensor components will also be normal.

As was demonstrated above, at the boundary of the plate $x_2 = 0$ the stress concentration factor according to mean square deviation considerably exceeds unity. Therefore, it is possible to assume that the main contribution to the reduction in reliability is given by boundary effects [12]. The reliability function for a plate may be determined approximately as the probability of a random occurrence, consisting of the fact that at the boundary coinciding with line $x_2 = 0$ there will not be a single stress σ^*_{11} outburst beyond the permissible level σ_*:

$$P(L) = P\left[\sup_{a \leqslant x_1 \leqslant b} \sigma^*_{11}(x_1) < \sigma_*\right] \tag{4.1}$$

($L = b - a$ is length of the plate boundary).

By assuming that the plate is a highly reliable structure, the probability of failure-free operation (4.1) is expressed approximately in terms of the average number of outbursts from a region of permissible values [12]

$$P(L) \approx \exp[-\lambda(\sigma_*)L]. \tag{4.2}$$

The average number of outbursts $\lambda(\sigma_*)$ arriving at a unit of length is found for normal function σ^*_{11} by an equation [12]

$$\lambda(\sigma_*) = \frac{\sqrt{D'_{11}(0)}}{2\pi\sqrt{D_{11}(0)}} \exp\left[-\frac{\sigma^2_*}{2D_{11}(0)}\right].$$

TABLE 1

α	$d_{11}=\sqrt{D_{11}(0)}/\sigma^0\cdot 100\%$							
	n							
	0	1	2	3	4	5	6	7
0,05	3,53	2,83	2,36	2,02	1,77	1,57	1,41	1,29
0,1	7,07	5,66	4,71	4,04	3,53	3,14	2,83	2,57
0,2	14,14	11,31	9,43	8,08	7,07	6,29	5,66	5,14
0,3	21,21	16,87	14,14	12,12	10,61	9,43	8,48	7,71
0,4	28,28	22,63	18,86	16,16	14,14	12,57	11,31	10,28
0,5	35,36	28,28	23,57	20,20	17,68	15,71	14,14	12,86

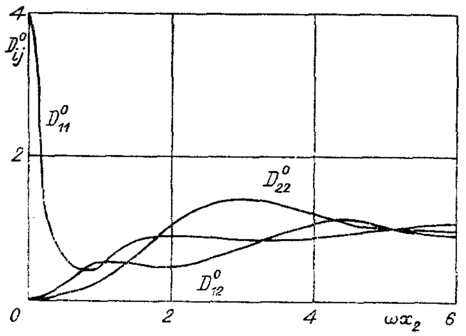

Fig. 1

Here $D_{11}(0)$ is dispersion of random function σ^*_{11} for $x_2 = 0$; $D'_{11}(0)$ is dispersion of derivative $\sigma^*_{1,1}$ with $x_2 = 0$. Dispersion $D'_{11}(0)$ is calculated by the equation

$$D'_{11}(0) = -\frac{d^2}{d\tau^2} K_{11}(\tau)|_{\tau=0},\ \tau = x_1^2 - x_1^1, \tag{4.3}$$

where the correlation function for stress σ^*_{11} with $x_2 = 0$

$$K_{11}(\tau) = \langle \overline{\sigma^*_{11}(x_1)}\,\sigma^*_{11}(x_1+\tau)\rangle = \frac{8\alpha^2\sigma^{02}}{(4+n)^2}\exp(i\omega\tau) \tag{4.4}$$

(the line indicates complex conjugation).

Taking account of Eqs. (3.1), (4.3), (4.4) in order to calculate the average number of outbursts $\lambda(\sigma_*)$ arriving at a unit length we obtain finally an expression

$$\lambda(\sigma_*) = \frac{\omega}{2\pi}\exp\left[-\frac{(4+n)^2\sigma_*^2}{16\alpha^2\sigma^{02}}\right]. \tag{4.5}$$

Thus, Eqs. (4.2) and (4.5) make it possible according to a prescribed permissible determinate level of σ_* to estimate plate reliability approximately.

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